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Stability analysis for the quartic Landau–Ginzburg model: II

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Abstract. A previous investigation of the stability of static one-dimensional solutions of the Landau–Ginzburg equation with a quartic non-linearity is extended. Exact spatially varying solutions are modified by small amplitude, time-dependent perturbations. In contradiction to the case of part I of this study, these are not assumed to have small frequency ω and small decay rates γ. We show that all periodic solutions, as well as the solitary waves, are unstable with respect to this new type of perturbations. The kink solution is stable with respect to all perturbations considered. When the results of both parts of this paper are put together, we obtain an extensive stability analysis of static solutions to the Landau–Ginzburg equation. This equation is important in fluid dynamics, solid state and superconductivity theory, as well as other branches of physics. The paper is self-contained and can be read independently of part I.

1. Introduction

We consider the Landau–Ginzburg equation in the form

$$(1/\Gamma)\partial M/\partial t = 2\nabla^2 M - V'(M) \qquad \Gamma > 0. \tag{1.1}$$

For applications and an interpretation of parameters in each case see, for example Khan (1986), Winternitz *et al* (1988), (magnetic phase transitions); Nabarro (1979), Löwen and Oxtoby (1990), Harrowell (1987), Dieterich (1990), Munakata (1990) (various solid state contexts including dynamical extensions of density functional theory); Chen and Whitehead (1968) (Bénard cell context); Infeld *et al* (1990), (nucleation of coherent structures); and Infeld and Rowlands (1990) (plasma physics).

Among the solutions obtained to (1.1) are spatially one-dimensional ones expressible in terms of Jacobi elliptic functions $sn(x, K)$, $cn(x, K)$ and $dn(x, K)$ and their limiting cases: solitary pulses and kinks for $K \to 1$ and trigonometric functions for $K \to 0$. In Grundland *et al* (1990), referred to as part I, we considered approximate solutions, obtained by adding to the non-linear waves small amplitude perturbations with wavelengths much larger than those of the non-linear waves. We then expanded in the wavenumber k of the perturbation and obtained the temporal behaviour. The

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method used is due to Infeld and Rowlands (1979) and is described in chapter 8 of Infeld and Rowlands (1990). For all cases considered, the frequency ω or growth rate γ were proportional to k^2. The non-linear cnoidal waves, as well as their solitary wave limit, were stable with respect to small k perturbations. The waves described by the *sn* and *dn* functions were unstable but those described by *cn* functions were stable. The kink solution, corresponding to Bloch domain walls in the magnetic phase transition context, was found to be marginally stable.

In the present analysis we will look at a different class of perturbation, having exactly the same parallel component of the wavevector as the basic non-linear wave. New instabilities appear and in fact only the kink solution survives both tests and so far seems to be stable.

The complete analysis (parts I and II) illustrates how complementary different stability calculations can be. One should be very cautious about concluding that a solution is stable just from one, limited calculation. On the other hand, our stability classification is now reasonably complete for the Landau–Ginzburg equations, as all solutions but one are seen to be unstable. In that sense the pursuit of further instabilities could add nothing new. However in a given experimental configuration one must check whether the instability found here (or one of them when there are several) can build up fast enough to destroy our non-linear structure.

2. Linearization in perturbed quantities

Suppose we have an x-dependent solution to (1.1), $M_0(x)$, and perturb it

$$M(\boldsymbol{x},t) = M_0(x) + \delta M(\boldsymbol{x},t). \tag{2.1}$$

If we linearize in δM we obtain, from (1.1)

$$-1/\Gamma(\delta M)_t + 2\nabla^2\delta M - V''(M_0)\delta M = 0. \tag{2.2}$$

Since the coefficients in (2.2) are independent of time and periodic in x, we use Floquet's theorem (Ince 1956) to write

$$\delta M(\boldsymbol{x},t) = \psi(x)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \gamma t) \tag{2.3}$$

where $\psi(x)$ is periodic in x with the same period as M_0. In part I, the modulus of the real wavenumber $k = ({k_x}^2 + {k_y}^2)^{1/2}$ was considered to be small, as was γ. Here in part II we take k_x to be zero and k_y arbitrary. The value of γ is not assumed to be small and in fact will follow from the analysis. For the moment we will keep k_x in the equations.

The function ψ satisfies

$$\begin{aligned} L\psi &= -(\gamma/\Gamma)\psi - 4\mathrm{i}k_x\partial\psi/\partial x + 2k^2\psi \\ L &= 2\frac{\partial^2}{\partial x^2} - V''(M_0). \end{aligned} \tag{2.4}$$

In what follows we will take †

$$V(M) = -cM^4 + 2bM^2 + a. \tag{2.5}$$

Equations (2.3), (2.4) and (2.5) will be the basis of our subsequent analysis.

† There is a misprint on page 7147 of part I, where minus this quantity appears (line 6 down). Also M^2 in equation (3.1) should be ${M_x}^2$, and M^2 in the figure caption on page 7148 should be ${M_x}^2$.

2.1. Constant M limit

One simple solution to (1.1) is the constant one such that

$$V'(M) = 0$$

given from (2.5) by

$$M_0^2 = b/c.$$

The stability analysis is particularly simple to perform in this limit and the result will help us understand things to come. We now look at perturbations such that $\psi(x)$ in (2.3) is constant. We obtain, from (2.3) and (2.4),

$$\gamma/\Gamma = -8b + 2k^2. \tag{2.6}$$

In much of what follows we will be concerned with $k = 0$ and will hope to recover the $\gamma/\Gamma = -8b$ root in the $M \to M_0$ limit.

2.2. Small k limit

In part I we found solutions such that $\gamma \sim k^2$ for all cases. Thus, in the zero k limit we expect $\gamma = 0$ to be a root for all cases.

To sum up the results of subsections 2.1 and 2.2: $\gamma = 0$ should appear as a root for all classes of non-linear wave and soliton; $\gamma = -8\Gamma b$ should appear as a small amplitude root for the class that contains $M = M_0$ (the case described by the dn function, figure 1(a) of part I).

2.3. Classification of exact solutions

Exact, one variable dependent solutions to (1.1) were classified in detail in part I. However, we will now repeat this classification very briefly. Details and appropriate phase space (M_x, M) diagrams can be found in part I.

Equation (1.1) in one variable x is integrated once to give (using (2.5))

$$M_x^2 = a + 2bM^2 - cM^4 \tag{2.7}$$

so roots of M_x^2 are located at

$$M_{1,2}^2 = \frac{b \pm (b^2 + ac)^{1/2}}{c}. \tag{2.8}$$

Meaningful solutions are always such that M is bounded by two (real) roots from among these four. The three classes of solutions are denoted by (I), (II) and (III) as follows:

(I) All four roots $\pm M_1$, $\pm M_2$ real and $c > 0$.

$$\begin{aligned} M &= \pm M_2 dn[c^{1/2} M_2 (x - x_0), q] \\ q^2 &= 1 - M_1^2/M_2^2. \end{aligned} \tag{2.9}$$

In the constant M limit considered earlier

$$M_{1,2}{}^2 = b/c \qquad \text{and} \qquad q^2 = 0.$$

In the soliton limit $q^2 \to 1$ we have

$$M = \pm M_2 \mathrm{sech}[c^{1/2} M_2 (x - x_0)] \qquad M_2 = 2b/c. \tag{2.10}$$

See figures 1(a) and (b), part I.

(II) Two real roots $\pm M_1$, and two imaginary ones $\pm \mathrm{i}|M_2|$, and $c > 0$. Now

$$\begin{aligned} M &= M_1 cn[\{c(M_1^2 + |M_2|^2)\}^{1/2}(x - x_0)q] \\ q^2 &= \frac{M_1^2}{M_1^2 + |M_2|^2} \qquad M_2 = \mathrm{i}|M_2| \neq 0. \end{aligned} \tag{2.11}$$

See figures 1(c) and (d) of part I for 0 and $b > 0$ respectively.

For $M_2 \to 0$, i.e. $q^2 \to 1$ we recover the solitary wave (2.10) again. For $q^2 \to 0$ we only obtain the trivial limit $M = 0$ (disordered phase).

(III) Four real roots $\pm M_{1,2}$ and $c < 0$.

$$\begin{aligned} M &= M_1\, sn[|c|^{1/2} M_2 (x - x_0), q] \\ q^2 &= (M_1/M_2)^2. \end{aligned} \tag{2.12}$$

In the $q^2 = 1$ limit we obtain the kink

$$M = \pm M_1 \tanh[|c|^{1/2} M_1 (x - x_0)] \tag{2.13}$$

whereas $q^2 \to 0$ corresponds to the disordered phase.

3. Stability analysis for $k_x = 0$

We will now perform a small perturbation stability analysis around the solutions given in section 2.3 assuming $k_x = 0$. The perpendicular component k_y can be arbitrary and γ/Γ for arbitrary k_y can be obtained from the result for $k_y = 0$ by shifting according to

$$\frac{\gamma(k_y)}{\Gamma} = \frac{\gamma(0)}{\Gamma} + 2k_y^2. \tag{3.1}$$

In what follows we will not bother about the k_y dependence other than to observe that non-zero k_y always stabilizes these modes, just as it did the modes of part I.

The basis of our analysis will be a consideration of the equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}u^2} + [\lambda - 6q^2 sn^2(u,q)]\psi = 0. \tag{3.2}$$

This is a special case of the Lamé equation, see Magnus and Winkler (1972). The eigenvalues and eigenfunctions of (3.2) will be related to those of (2.4) with $k_x = 0$ and M_0 given by the formulas of section 2.3 one after the other. We have $u =$

Table 1. Eigenfunctions and eigenvalues of the Lamé equation (3.2). The arguments and modules of the Jacobi elliptic functions are the same as in (3.2).

ψ_n	$\lambda_n(q)$;	$\lambda_n(0)$;	$\lambda_n(1)$;
$\psi_1 = sn^2 - 2/\lambda_1$	$2[1+q^2+r]$ $r=(q^4-q^2+1)^{1/2}$	4	6
$\psi_2 = sn\,cn$	$4+q^2$	4	5
$\psi_3 = sn\,dn$	$1+4q^2$	1	5
$\psi_4 = cn\,dn$	$1+q^2$	1	2
$\psi_5 = sn^2 - 2/\lambda_5$	$2[(1+q^2)-r]$	0	2

$u_0(x-x_0)$, u_0 given in (2.9)–(2.12), respectively. Patera and Winternitz (1973, 1976) have shown that there are exactly five eigenvalues of (3.2) associated with five bounded eigenfunctions ψ_n (for early work, covering special cases only, see Infeld and Hull (1951)). The other five eigenfunctions are unbounded in x and therefore of no interest here. The bounded eigenfunctions are given in the Patera and Winternitz references. However, we now give them in different form, more easily verifiable in our context, and order them so that λ_n decreases with n. They are shown in table 1.

Eigenvalues never cross and they merge two by two in the two limits $q=0$ and $q=1$ to form a 'coiled snake' figure. An illustration of this kind of behaviour can be found in Patera and Winternitz (1976, figure 1). The first and last eigenfunctions could have been found by looking for (const $+ M^2$) eigenfunctions without even using the form of the solution! We see from table 1 that the solution ψ_5 is singular in the limit $q=0$. In this limit (3.2) reduces to

$$\mathrm{d}^2\psi_5/\mathrm{d}u^2 = 0.$$

Since ψ_5 is linear in u, a secular term appears in general. The constant solution is valid.

Table 2. Growth rates for dn-waves; r as in table 1.

$\tilde\gamma = \gamma/[2\Gamma c M_2^2]$	$\tilde\gamma(q=0)$; constant M_0	$\tilde\gamma(q^2=1)$; soliton
q^2-2+2r	0	1
0	0	0
$3(q^2-1)$	−3	0
−3	−3	−3
q^2-2-2r	−4	−3

Table 3. Growth rates for cn-waves; r as in table 1.

$\tilde\gamma = \gamma/[2\Gamma c M_1^2 + \lvert M_2\rvert^2]$	$\tilde\gamma(q^2=0)$	$\tilde\gamma(q^2=1)$; soliton
$-2q^2+1+2r$	3	1
$3(1-q^2)$	3	0
0	0	0
$-3q^2$	0	−3
$-2q^2+1-2r$	−1	−3

Table 4. Growth rates for *sn*-waves; r as in table 1.

$\bar{\gamma}=\gamma/[2\Gamma\|c\|M_2^2]$	$\bar{\gamma}(q^2=0)$	$\bar{\gamma}(q^2=1)$; kink
$1+q^2+2r$	3	4
3	3	3
$3q^2$	0	3
0	0	0
$1+q^2=2r$	−1	0

We now write (3.2) in two other, equivalent forms using the identities $sn^2+cn^2=1$ and $dn^2+q^2sn^2=1$:

$$d^2\psi/du^2+[\lambda-6q^2+6q^2cn^2(u,q)]\psi=0 \tag{3.3}$$

$$d^2\psi/du^2+[\lambda-6+6dn^2(u,q)]\psi=0 \tag{3.4}$$

and use the information given in table 1 to solve (2.4) for the three principal forms of M_0. We obtain , after dropping x_0, the following results.

Case 1. $M=M_2dn[c^{1/2}M_2x,q]$. The growth rates are summed up in table 2.

We recover the constant M limit, $M^2=b/c$, $\gamma/\Gamma=-8b$, as well as the small γ,k case, which for our purposes is $\gamma=0$ throughout (second case). The other three roots (first, third, fourth) are completely new. As negative γ means instability, we have two new unstable modes. All case 1 non-linear waves and the soliton are unstable.

Case 2. $M=M_1\,cn[c(M_1^2+|(M_2)|^2)^{1/2}x,q]$. The growth rates are summed up in table 3.

The soliton limit is common to cases 1 and 2 and this gives a check on the results. All waves and also the soliton limit are unstable. All statements for $n=5$ in the $q=0$ limit must be taken with a grain of salt, since this is the one and only case where secular terms are present in the perturbative expansion, invalidating the very expansion used.

Case 3. $M=M_1sn[|c|^{1/2}M_2x,q]$. The growth rates are summed up in table 4.

The values at the $q=0$ limit are common with those of case 2. Values at the $q^2=1$ limit are minus those for the $q^2=0$ limit of case 1. All non-linear waves are unstable, though the number of unstable perturbed modes has decreased to one. The kink, however, is stable. It is the sole 'survivor' of this analysis in terms of stability. This is important, as the kink solution is significant in magnetic phase transition considerations (as well as other situations in which we have a transition between two distinct regions.) Inclusion of k_x^2 corrections further stabilizes the kink (this is always true of k_y^2 corrections as already seen).

We see that for the dn solution we have obtained three unstable modes $n=(3,4,5)$, for cn two (4,5), and for sn one (5). (The analysis does not apply for $n=5$, $q=0$, as secular terms are present in the perturbation expansion.) All non-linear periodic solutions are unstable with respect to the perturbation considered here, as are the solitary waves.

4. Summary of parts I and II

We have performed two complementary stability calculations. In part I some of the non-linear waves were seen to be unstable. In part II it was shown that among all the nonzero static solutions of the Landau–Ginzburg equation (1.1) with potential (2.5), only the kink is stable with respect to all time-dependent perturbations considered. This in itself is important, since the kink corresponds to domain walls and is hence significant in many physical studies. Whether any of the exact solutions are observable in concrete physical settings depends primarily on the type of perturbations that occur. In any case, the indication is that we are dealing with transient phenomena corresponding to exact static solutions. The rate at which they will disappear depends on the numerical values of the growth rate γ in each specific physical situation.

Recently Löwen and Oxtoby (1990) looked at a form of the non-linear Landau–Gingzburg equation where two parabolas were taken to model the non-linear term. This makes it possible to obtain exact solutions corresponding to the relaxation of step function distributions to the 'kink' profile. In our analysis the kink is stable and is also expected to be so for the two-parabola $V(M)$. Otherwise the two calculations (Löwen and Oxtoby's and ours) are different but somewhat complementary.

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